Fall 2017 : Problem 2 : Suppose that (X,Y) has the characteristic function $\varphi_{(X,Y)}(s,t) = \exp\{\alpha(e^{is}-1)+\beta(e^{it}-1)+\gamma(e^{i(s+t)}-1)\}$, with $\alpha>0, \beta>0$ and $\gamma>0$.

(a.) To show : X and Y are Poisson distributed, but $X+Y$ is not.

$\varphi_X(s) = \mathbb{E}[e^{isX}] = \mathbb{E}[e^{isX+i\cdot 0\cdot Y}] = \varphi_{(X,Y)}(s,0) = \exp\{\alpha(e^{is}-1)+\gamma(e^{is}-1)\} = \exp\{(\alpha+\gamma)(e^{is}-1)\}$

Reminder : $\tilde{X} \in Po(\lambda)$ with $\lambda>0 \Leftrightarrow \varphi_{\tilde{X}}(s) = \exp\{\lambda(e^{is}-1)\}$, $\forall s\in\mathbb{R}$.

$\Rightarrow X \in Po(\alpha+\gamma)$.

$\varphi_Y(t) = \mathbb{E}[e^{itY}] = \mathbb{E}[e^{i0X+itY}] = \varphi_{(X,Y)}(0,t) = \exp\{\beta(e^{it}-1)+\gamma(e^{it}-1)\} = \exp\{(\beta+\gamma)(e^{it}-1)\}$.

$\Rightarrow Y \in Po(\beta+\gamma)$

$\varphi_{X+Y}(s) = \mathbb{E}[e^{isX+isY}] = \varphi_{(X,Y)}(s,s) = \exp\{\alpha(e^{is}-1)+\beta(e^{is}-1)+\gamma(e^{i2s}-1)\}$
$= \exp\{(\alpha+\beta)(e^{is}-1)+\gamma(e^{i2s}-1)\}$

Since $\forall\lambda\in\mathbb{R}\ \exists s\in\mathbb{R}$; $(e^{is}-1)\neq\lambda(e^{2is}-1)$, $X+Y$ is not Poisson-distributed.

(b.) Are X and Y independent ?

No, since $\varphi_X(s)\varphi_Y(t) = \exp\{\alpha(e^{is}-1)+\beta(e^{it}-1)+\gamma((e^{is}-1)+(e^{it}-1))\} \neq \varphi_{(X,Y)}(s,t)$

Fall 2017 : Problem 3 : Let $(X_i)_{i\geq 1}$ be iid $U(0,1)$-distributed random variables and $N\in Po(\lambda)$ be independent of $(X_i)_{i\geq1}$. Set $Y_N := \max\{X_1,\dots,X_N\}$ when $N>0$
$Y_0 = 0$ otherwise

(a.) Compute the distribution and the characteristic functions of Y_N :

Fix $u\in[0,1]$. $\mathbb{P}(Y_N\leq u) = \sum_{k\geq0}\mathbb{P}(Y_N\leq u \text{ and } N=k) \overset{N \text{ ind of } (X_i)_{i\geq1}}{=} \sum_{k\geq0}\mathbb{P}(Y_k\leq u)\mathbb{P}(N=k)$

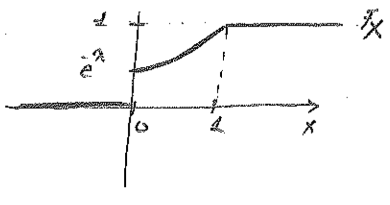
$= \mathbb{P}(N=0) + \sum_{k\geq1}\mathbb{P}(\max\{X_1,\dots,X_k\}\leq u)\,\mathbb{P}(N=k) = \mathbb{P}(N=0)+\sum_{k\geq1}\mathbb{P}(\bigcap_{i=1}^{k}\{X_i\leq u\})\,\mathbb{P}(N=k)$

$\max\{X_1,\dots,X_k\}\leq u \Leftrightarrow \bigcap_{i=1}^{k}\{X_i\leq u\}$

$\overset{(X_i)_{i\geq1} \text{ iid}}{=} \mathbb{P}(N=0)+\sum_{k\geq1}\mathbb{P}(X_1\leq u)^k\,\mathbb{P}(N=k) = e^{-\lambda}+\sum_{k\geq1}u^k\frac{e^{-\lambda}\lambda^k}{k!} = e^{-\lambda}\left(1+\sum_{k\geq1}\frac{(u\lambda)^k}{k!}\right) = e^{-\lambda}e^{u\lambda} = e^{-\lambda(1-u)}$

Notice that when $u=0$, $\mathbb{P}(Y_N\leq0) = e^{-\lambda}>0$ and since $Y_N\geq0$, $\mathbb{P}(Y_N\leq0) = \mathbb{P}(Y_N=0)$
$\Rightarrow \mathbb{P}(Y_N=0) = e^{-\lambda}>0$.

So if we draw the graph of $x\longmapsto F_{Y_N}(x)$, we see that we have a discontinuity at 0!



We must be careful.

on $(0,1)$, F_{Y_N} is continuous (even differentiable!) so we can compute the p.d.f of Y_N.

$$f_{Y_N}(u) = \frac{d}{du}\mathbb{P}(Y_N \le u) = \frac{d}{du} e^{-\lambda(1-u)} = \lambda e^{-\lambda(1-u)}$$

Now $$\varphi_{Y_N}(t) = \mathbb{E}[e^{itY_N}] = \mathbb{E}[e^{itY_N}\mathbb{1}_{\{Y_N=0\}}] + \mathbb{E}[e^{itY_N}\mathbb{1}_{\{Y_N>0\}}] = \mathbb{P}(Y_N=0) + \int_0^1 e^{itu} f_{Y_N}(u)\,du$$

$$= e^{-\lambda} + \int_0^1 e^{itu}\lambda e^{-\lambda(1-u)}\,du = e^{-\lambda} + \int_0^1 \lambda e^{itu-\lambda(1-u)}\,du = e^{-\lambda} + \frac{\lambda e^{itu-\lambda(1-u)}}{it+\lambda}\Big|_0^1 = e^{-\lambda} + \frac{\lambda e^{it}}{it+\lambda} - \frac{\lambda e^{-\lambda}}{it+\lambda}$$

$$= \underline{\underline{e^{-\lambda} + \frac{\lambda}{it+\lambda}\left(e^{it} - e^{-\lambda}\right)}}$$

(b.) To show: $\mathbb{E}[Y_N] \to 1$ as $\lambda \to +\infty$.

$$\mathbb{E}[Y_N] = \int_0^{+\infty}\mathbb{P}(Y_N>t)\,dt \underset{Y_N\in[0,1]}{=} \int_0^1 \mathbb{P}(Y_N>t)\,dt = \int_0^1 (1-\mathbb{P}(Y_N\le t))\,dt = 1 - \int_0^1 e^{-\lambda(1-t)}\,dt = 1 - \frac{1}{\lambda}e^{-\lambda(1-t)}\Big|_0^1 = \underbrace{1-\frac{1}{\lambda}(1-e^{-\lambda})}_{\to\, 1 \text{ as } \lambda\to+\infty.}$$

(c.) To show: $\lambda(1-Y_N)$ converges in distribution as $\lambda \to +\infty$.

If $Y_N\in[0,1]$, then $\lambda(1-Y_N)\in[0,\lambda]$. Fix $u\in[0,\lambda)$,

$$\mathbb{P}(\lambda(1-Y_N)\le u) = \mathbb{P}\left(Y_N \ge 1-\frac{u}{\lambda}\right) = 1-\mathbb{P}\left(Y_N < 1-\frac{u}{\lambda}\right) = 1-e^{-\lambda(1-(1-\frac{u}{\lambda}))} = 1-e^{-u}$$

$$\Rightarrow F_{\lambda(1-Y_N)}(u) = \begin{cases} 0 & \text{if } u<0 \\ 1-e^{-u} & \text{if } u\in[0,\lambda) \\ 1 & \text{if } u\ge\lambda \end{cases} \qquad \Rightarrow Y_N \xrightarrow{d} Exp(1) \text{ as } \lambda\to+\infty.$$

QED.

Fall 2016: Problem 3: Let X, Y, Z be iid $\mathcal{N}(0,1)$-distributed random variables.

(a.) Determine the distributions of $U = X+Y+Z$
$V = 2X-Y-Z$
$W = Y-Z$

1.) Notice that $\begin{pmatrix}X\\Y\\Z\end{pmatrix}$ is a Gaussian vector.

Indeed $\alpha X+\beta Y+\gamma Z$ is a normal random variable $\forall \alpha,\beta,\gamma\in\mathbb{R}$ since X,Y,Z are independent, and by the course, $\left\{\alpha X+\beta Y+\gamma Z \text{ is normal } \forall\alpha,\beta,\gamma\in\mathbb{R}\right\} \overset{(*)}{\Leftrightarrow} \left\{\begin{pmatrix}X\\Y\\Z\end{pmatrix} \text{ is a Gaussian vector}\right\}$.

2.) Notice that $\begin{pmatrix}U\\V\\W\end{pmatrix}$ is a Gaussian vector. Let $\alpha,\beta,\gamma\in\mathbb{R}$.

$\alpha U+\beta V+\gamma W = \alpha(X+Y+Z)+\beta(2X-Y-Z)+\gamma(Y-Z) = (\alpha+2\beta)X+(\alpha-\beta+\gamma)Y+(\alpha-\beta-\gamma)Z$, which is normal since $\begin{pmatrix}X\\Y\\Z\end{pmatrix}$ is a Gaussian vector. $\overset{(*)}{\Rightarrow} \begin{pmatrix}U\\V\\W\end{pmatrix}$ is a Gaussian vector.

We just need to compute the expectation and variance of U, V, W to determine their distribution, since they are normally distributed.

$\mathbb{E}[U] = \mathbb{E}[X] + \mathbb{E}[Y] + \mathbb{E}[Z] = 0$

$\mathbb{E}[V] = 2\mathbb{E}[X] - \mathbb{E}[Y] - \mathbb{E}[Z] = 0$

$\mathbb{E}[W] = \mathbb{E}[Y] - \mathbb{E}[Z] = 0.$

$Var(U) = Var(X) + Var(Y) + Var(Z) = 3$

X, Y, Z independent

$Var(V) = Var(2X) + Var(-Y) + Var(-Z) = 4\,Var(X) + Var(Y) + Var(Z) = 6$

$Var(W) = Var(Y) + Var(-Z) = Var(Y) + Var(Z) = 2.$

$\Rightarrow U \in \mathcal{N}(0,6),\ V \in \mathcal{N}(0,3)$ and $W \in \mathcal{N}(0,2)$.

(b) Determine the distribution of the random vector $\begin{pmatrix} U \\ V \\ W \end{pmatrix}$.

$\begin{pmatrix} U \\ V \\ W \end{pmatrix}$ is Gaussian $\Rightarrow$ We just have to find its mean and covariance matrix to determine its distribution.

$\mathbb{E}\left[\begin{pmatrix} U \\ V \\ W \end{pmatrix}\right] = \begin{pmatrix} \mathbb{E}[U] \\ \mathbb{E}[V] \\ \mathbb{E}[W] \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ from before

X, Y, Z independent $\mathcal{N}(0,1)$

$$Cov\left(\begin{pmatrix} U \\ V \\ W \end{pmatrix}\right) = Cov\left(\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}\right) = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & 1 & -1 \end{pmatrix} Cov\left(\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}\right)\begin{pmatrix} 1 & 2 & 0 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 0 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$ $\Rightarrow U, V, W$ are independent $\mathcal{N}(0,6), \mathcal{N}(0,3), \mathcal{N}(0,2)$ - distributed random variables respectively.

(c) Determine $\mathbb{E}[W^2U^2 + V^2U^3 \mid U=3] = \mathbb{E}[W^2 \cdot 9 + V^2 \cdot 27 \mid U=3] \overset{ind}{=} \mathbb{E}[W^2 \cdot 9 + V^2 \cdot 27] = 9 \cdot \mathbb{E}[W^2] + 27\,\mathbb{E}[V^2]$

$= 9\,Var(W) + 27\,Var(V) = 9 \cdot 2 + 27 \cdot 6 = \underline{\underline{180}}$

$\mathbb{E}[W] = \mathbb{E}[V] = 0$

(d) To show: $X-Y$, $Y-Z$ and $Z-X$ are independent of U.

Since $X - Y = \frac{1}{2}(V - W)$
$Y - Z = W$
$Z - X = -\frac{1}{2}(V + W)$

and U, V, W are independent, then $X-Y$, $Y-Z$ and $Z-X$ are independent of U.

Fall 2017 : Problem 4 : Let $(X_i)_{i\geq 1}$ be iid with mean μ and variance σ^2.

Let $\bar{X}_n := \frac{1}{n}\sum_{i=1}^{n} X_i$

(a.) To show: $\sum_{i=1}^{n}(X_i-\bar{X}_n)^2 = \sum_{i=1}^{n}(X_i-\mu)^2 - n(\bar{X}_n-\mu)^2$

$$\sum_{i=1}^{n}(X_i-\bar{X}_n)^2 = \sum_{i=1}^{n}(X_i-\mu+\mu-\bar{X}_n)^2 = \sum_{i=1}^{n}((X_i-\mu)-(\bar{X}_n-\mu))^2 = \sum_{i=1}^{n}(X_i-\mu)^2 + \sum_{i=1}^{n}(\bar{X}_n-\mu)^2 - 2\sum_{i=1}^{n}(X_i-\mu)(\bar{X}_n-\mu)$$

$$= \sum_{i=1}^{n}(X_i-\mu)^2 + n(\bar{X}_n-\mu)^2 - 2(\bar{X}_n-\mu)\cdot n(\bar{X}_n-\mu) = \sum_{i=1}^{n}(X_i-\mu)^2 - n(\bar{X}_n-\mu)^2$$

ok!

(b.) Let $S_n^2 := \frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X}_n)^2$. To show: $S_n \xrightarrow{\mathbb{P}} \sigma$, as $n\to+\infty$.

Remmder: Let $(X_n)_{n\geq 1}$ and X be such that $X_n \xrightarrow{\mathbb{P}} X$ as $n\to+\infty$, and let $\begin{matrix}\mathbb{R}\to\mathbb{R}\\ x\mapsto g(x)\end{matrix}$ be continuous.
Then $g(X_n)\xrightarrow{\mathbb{P}} g(X)$ as $n\to+\infty$.

1./ Work with S_n^2 : $S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X}_n)^2 \overset{(a.)}{=} \frac{1}{n-1}\left\{\sum_{i=1}^{n}(X_i-\mu)^2 - n(\bar{X}_n-\mu)^2\right\}$

$$= \frac{n}{n-1}\cdot\frac{1}{n}\left\{\sum_{i=1}^{n}(X_i-\mu)^2 - n(\bar{X}_n-\mu)^2\right\} = \frac{n}{n-1}\left(\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2 - (\bar{X}_n-\mu)^2\right)$$

LLN: $\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2 \xrightarrow{\mathbb{P}} \mathbb{E}[(X_1-\mu)^2] = Var(X_1) = \sigma^2$

LLN: $(\bar{X}_n-\mu) = \frac{1}{n}\sum_{i=1}^{n}X_i - \mu \xrightarrow{\mathbb{P}} 0$

By the remmder, $(\bar{X}_n-\mu)^2 \xrightarrow{\mathbb{P}} 0$ as $n\to+\infty$

Since $X_n\xrightarrow{\mathbb{P}} X$ and $Y_n\xrightarrow{\mathbb{P}} Y \Rightarrow X_n - Y_n \xrightarrow{\mathbb{P}} X-Y$ as $n\to+\infty$, and since $\frac{n}{n-1}\to 1$ as $n\to+\infty$,

$S_n^2 \xrightarrow{\mathbb{P}} \sigma^2$ as $n\to+\infty$.

By the remmder, since $x\mapsto\sqrt{x}$ is continuous, $S_n = \sqrt{S_n^2} \xrightarrow{\mathbb{P}} \sqrt{\sigma^2} = \sigma$ as $n\to+\infty$.

QED

(c.) Find the limiting distribution of $\frac{\sqrt{n}(\bar{X}_n-\mu)}{S_n}$ as $n\to+\infty$.

$\sqrt{n}(\bar{X}_n-\mu) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(X_i-\mu) \xrightarrow{d} \mathcal{N}(0,\sigma^2)$. by the central limit theorem.

Now by (b.), $S_n \xrightarrow{\mathbb{P}} \sigma$ as $n\to+\infty$.

By slutsky, $\frac{\sqrt{n}(\bar{X}_n-\mu)}{S_n} \xrightarrow{d} \frac{\mathcal{N}(0,\sigma^2)}{\sigma} = \underline{\mathcal{N}(0,1)}$

QED